Musterlösung Probeklausur

1) Horner

	1	0	-3	1	-1
$x_0 = 3$	X	3	9	18	57
	1	3	6	19	56

$\underline{56}$ $\uparrow$ $f(3)$

a) $f(3) = 56$

b) $g(x) = 1 \cdot x^3 + 3x^2 + 6x + 19 + \frac{56}{x-3}$

c) $\underbrace{x^4 - 9x^3 + 25x^2 - 27x + 10}_{f(x)} : (x-1) = x^3 - 8x^2 + 17x - 10$

$$
\begin{array}{l}
-(x^4 - x^3) \\
\hline
\quad -8x^3 + 25x^2 \\
\quad -(-8x^3 + 8x^2) \\
\hline
\qquad 17x^2 - 27x \\
\qquad -(17x^2 - 17x) \\
\hline
\qquad\quad -10x + 10 \\
\qquad\quad -(-10x + 10) \\
\hline
\qquad\qquad 0
\end{array}
$$

$x^3 - 8x^2 + 17x - 10 : (x-1) = \underline{x^2 - 7x + 10}$

$$
\begin{array}{l}
-(x^3 - x^2) \\
\hline
\quad -7x^2 + 17x \\
\quad -(-7x^2 + 7x) \\
\hline
\qquad 10x - 10 \\
\qquad -(10x - 10) \\
\hline
\qquad\quad 0
\end{array}
$$

$x^2 - 7x + 10 = 0$

$x_{1,2} = \frac{7}{2} \pm \sqrt{\frac{49}{4} - 10}$

$\Rightarrow x_1 = 5$

$x_2 = 2$

$\Rightarrow f(x) = (x-1)^2 \cdot (x-5)(x-2)$

3) $f(x) = a_0 + a_1(x-1) + a_2(x-1)(x-3) + a_3(x-1)(x-3)(x-4)$

$$
\begin{array}{cccccccc}
x_i & f(x_i) & & & & & & \\
1 & \boxed{0} = a_0 & & & & & & \\
 & & \frac{2-0}{3-1} = \boxed{1} = a_1 & & & & & \\
3 & 2 & & \frac{1-1}{4-1} = \boxed{0} = a_2 & & & & \\
 & & \frac{3-2}{4-3} = 1 & & \frac{-\frac{1}{2}-0}{7-1} = \boxed{-\frac{1}{12}} = a_3 & & & \\
4 & 3 & & \frac{-1-1}{7-3} = -\frac{1}{2} & & & & \\
 & & \frac{0-3}{7-4} = -1 & & & & & \\
7 & 0 & & & & & &
\end{array}
$$

$\Rightarrow f(x) = 0 + (x-1) + 0 \cdot (x-1)(x-3) + \left(-\frac{1}{12}\right) \cdot (x-1)(x-3)(x-4)$

$= x-1 - \frac{1}{12}(x-1)(x-3)(x-4)$

4) $\sqrt{2-2x} = x-1 \quad | \; (\;)^2$ - keine Äquivalenz

$2-2x \quad (x-1)^2 \Leftrightarrow x^2-2x+1 = 2-2x$

$\Leftrightarrow x^2-1 = 0 \quad \Leftrightarrow x=1$ oder $x=-1$

Probe: $x=1$: $\sqrt{2-2} = 0 \qquad 1-1 = 0$ ✓

$x=-1$: $\sqrt{2+2} = \sqrt{4} = 2 \qquad -1-1 = -2$ f

$\Rightarrow L = \{x \mid x=1\} = \{1\}$

5) $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ P3

$f(x) = 2x + 3$ $$f'(x_0) = \lim_{h \to 0} \frac{2\cdot(x_0+h)+3 - (2\cdot x_0 + 3)}{h}$$

$$= \lim_{h \to 0} \frac{2x_0 + 2h + 3 - 2x_0 - 3}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$$

6) a) $f(x) = \underbrace{x^2 \cos(x)}_{\text{Prod.regel}} + e^x \Rightarrow f'(x) = 2x\cos(x) - x^2 \sin(x) + e^x$

b) $f(x) = \sqrt{2x-4}$ $u(x) = 2x-4$

$u' = 2$ $f(u) = \sqrt{u}$ $f'(u) = \frac{1}{2\sqrt{u}}$

$$f'(u(x)) = \frac{1}{2\cdot\sqrt{2x-4}}$$

$$\Rightarrow f'(x) = 2 \cdot \frac{1}{2\sqrt{2x-4}} = \frac{1}{\sqrt{2x-4}}$$

c) $f(x) = 2^x \cdot \ln(x) \Rightarrow f'(x) = 2^x \cdot \frac{1}{x} + \ln(2) 2^x \cdot \ln(x)$

Bem.: $(2^x)' = \ln(2) \cdot 2^x$

d) $f(x) = 2^{2x+1} = 2^{2x} \cdot 2^1 = 2 \cdot 2^{2x} = 2 \cdot (2^2)^x = 2 \cdot 4^x$

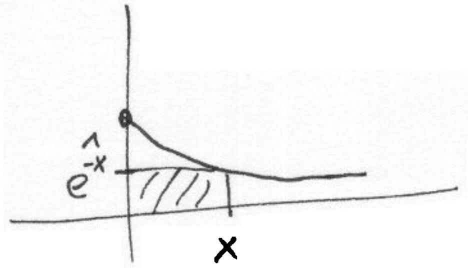
$\Rightarrow f'(x) = 2 \cdot \ln(4) \cdot 4^x$

e.) $f(x) = \ln\left(\frac{ax^2}{4+b}\right) = \ln(ax^2) + \ln(4+b) = \ln(a) + \ln(x^2) + \ln(4+b)$

$= \ln(a) + 2\ln(x) + \ln(4+b)$

$\rightarrow f'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$

7)



Fläche

$F(x) = x e^{-x}$

Optimum: $F'(x)=0$ $\quad F'(x) = e^{-x} - x e^{-x} = (1-x) e^{-x} = 0$

$\Rightarrow 1-x=0 \Rightarrow x=1$

$e^{-x}>0$

$F''(x) = -(1-x)e^{-x} - e^{-x} \Rightarrow F''(1) = -e^{-1} = -\frac{1}{e} < 0$

$\Rightarrow$ lok. Maximum

Fläche $F(1) = 1 \cdot e^{-1} = \underline{\underline{\frac{1}{e}}}$

8) $h = 0{,}1 \qquad f(x) = x e^{x} \qquad x_0 = 0$

rechtsseitig: $f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h} = \frac{f(0{,}1) - f(0)}{0{,}1}$

$= \frac{0{,}1 \cdot e^{0{,}1} - 0 \cdot e^{0}}{0{,}1} = e^{0{,}1} = \underline{\underline{1{,}105}}$

zentral: $f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} = \frac{f(0{,}1) - f(-0{,}1)}{0{,}2}$

$= \frac{0{,}1 \cdot e^{0{,}1} - (-0{,}1) e^{-0{,}1}}{0{,}2} = \frac{1}{2}\left(e^{0{,}1} + e^{-0{,}1}\right)$

$= \frac{1}{2}\left(1{,}105 + 0{,}905\right) = \frac{1}{2} \; 2{,}01 = \underline{\underline{1{,}005}}$

exakt: $f'(x) = x e^{x} + e^{x} \Rightarrow f'(0) = \underline{\underline{1}}$

9) $f(x) = x \cos(x) + x$

Taylorentw. um $x_0 = 0$

$$f_2(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2} f''(0)$$

$$f(0) = 0 \cdot 1 + 0 = 0$$

$$f'(x) = -x \sin(x) + \cos(x) + 2x \Rightarrow f'(0) = 1$$

$$f''(x) = -x\cos(x) - \sin(x) \overset{-\sin(x)}{\cancel{+\ldots}} + 2 \Rightarrow f''(0) = 2$$

$$\Rightarrow f_2(x) = 0 + 1 \cdot x + \frac{x^2}{2} \cdot 2 = \underline{\underline{x^2 + x}}$$

10) $e^x = 2 \iff \underbrace{e^x - 2}_{f(x)} = 0 \qquad f'(x) = e^x$

Newton

$$x_{n+1} = x_n - \frac{\overbrace{e^{x_n} - 2}^{f(x_n)}}{\underbrace{e^{x_n}}_{f'(x_n)}}$$

n	x_n	e^{x_n}	$x_{n+1} = x_n - \frac{e^{x_n} - 2}{e^{x_n}}$
0	0	1	$0 - \frac{1-2}{1} = 1$
1	1	2,71	$1 - \frac{0{,}71}{2{,}71} = 0{,}738$
2	0,738	2,092	$0{,}738 - \frac{0{,}092}{2{,}092} = 0{,}694$
3	0,694	2,002	$0{,}694 - \frac{0{,}002}{2{,}002} = 0{,}694$
4	$\underline{\underline{0{,}694}}$	×	×

11| $f(x)=3x+5$ $\qquad f'(x)=3$

$$O_0^2(3x+5)=\int_0^2 2\pi\,(3x+5)\cdot\sqrt{1+3^2}\,dx$$

$$=\int_0^2 2\pi\sqrt{10}\cdot(3x+5)\,dx=2\pi\sqrt{10}\cdot\left[3\frac{x^2}{2}+5x\right]$$

$$=2\pi\sqrt{10}\cdot(6+10)=32\pi\cdot\sqrt{10}\approx 317{,}9$$

12| $n=4 \Rightarrow h=\frac{1-0}{4}=0{,}25$

i	x_i	$f(x_i)=\frac{1}{x_i^3+1}$
0	0	1
1	0,25	0,985
2	0,5	0,889
3	0,75	0,703
4	1	0,5

$$\Rightarrow \int_0^1 \frac{1}{x^3+1}\,dx \approx 0{,}25\cdot\left(\frac{1+0{,}5}{2}+0{,}985+0{,}889+0{,}703\right)$$

$$=\underline{\underline{0{,}83175}}$$

3|

$$V = \left| \vec{F_1} \bullet (\vec{F_2} \times \vec{F_3}) \right|$$

$$\vec{F_2} \times \vec{F_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$$

$$\vec{F_1} \bullet \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} = 2 + 6 + 4 = \underline{\underline{12}}$$

$\Rightarrow \underline{\underline{V = 12}}$ Das Volumen des Spates beträgt 12 Einh.

14 mit $\begin{pmatrix}1\\0\\-1\end{pmatrix} + \alpha \begin{pmatrix}0\\1\\2\end{pmatrix} + \beta \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\\lambda\\2\end{pmatrix}$

Variante

a) $\Rightarrow \vec{n} = \begin{pmatrix}-2\\2\\-1\end{pmatrix} \Rightarrow$ Ebenengleichung

$$-2x + 2y - z = D$$

$$\text{mit } D = \begin{pmatrix}-2\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} = 1$$

$$\Rightarrow -2x + 2y - z = -1$$

Einsetzen $\Rightarrow -2 + 2\lambda - 2 = -1$

$$2\lambda = 3$$
$$\lambda = \frac{3}{2}$$

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b)

$1 + \beta = 1 \Rightarrow \beta = 0$

$\alpha = \lambda$

$-1 + 2\alpha = 2 \Rightarrow 2\alpha = 3 \Rightarrow \alpha = \frac{3}{2} \Rightarrow \lambda = \frac{3}{2}$

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5 a) Normale $\vec{n} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 8 \end{pmatrix}$; $\|\vec{n}\| = \sqrt{16+16+64} = \sqrt{96}$

Ebenengleichung $\vec{n} \cdot \vec{x} = \vec{n} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Leftrightarrow -4x+4y+8z = \begin{pmatrix} -4 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

$$\Leftrightarrow -4x+4y+8z = 8$$

$$d = \frac{\left| \begin{pmatrix} -4 \\ 4 \\ 8 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right) \right|}{\sqrt{96}} = \frac{|4-12|}{\sqrt{96}} = \frac{8}{\sqrt{96}} = \sqrt{\frac{64}{96}} = \underline{\underline{\sqrt{\frac{2}{3}}}}$$

b) $$\begin{pmatrix}1\\3\\0\end{pmatrix}+\alpha_1\begin{pmatrix}2\\0\\1\end{pmatrix}+\beta_1\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}2\\-2\\0\end{pmatrix}+\alpha_2\begin{pmatrix}-2\\0\\5\end{pmatrix}+\beta_2\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$2\alpha_1+2\alpha_2-\beta_2=1$$

$$4\beta_1+\beta_2=-5$$

$$\alpha_1+2\beta_1-5\alpha_2=0$$

$$\Rightarrow \left(\begin{array}{cccc|c}2&0&2&-1&1\\0&4&0&1&-5\\1&2&-5&0&0\end{array}\right) \qquad \left(\begin{array}{cccc|c}2&0&2&-1&1\\0&4&0&1&-5\\0&2&-6&\frac{1}{2}&-\frac{1}{2}\end{array}\right)$$

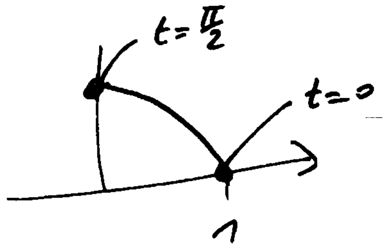
$$\beta_2=\lambda$$

$$\rightarrow \left(\begin{array}{cccc|c}2&0&2&&1\\0&4&0&1&-5\\0&0&12&0&-4\end{array}\right) \Rightarrow 12\alpha_2=-4 \rightarrow \alpha_2=-\frac{1}{3}$$

$\xrightarrow{\text{in } E_2}$: Schnitt: $$\vec{g}=\begin{pmatrix}2\\-2\\0\end{pmatrix}-\frac{1}{3}\begin{pmatrix}-2\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$=\begin{pmatrix}8/3\\-2\\-\frac{5}{3}\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

16| $\vec{x}(t) \quad \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$



$\vec{F}(x,y) = \begin{pmatrix} x^2+y^2 \\ 2 \end{pmatrix} \qquad \vec{x}\,'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$

$$\Rightarrow W = \int_0^{\frac{\pi}{2}} \vec{F}\left(\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}\right) \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt$$

$$\int_0^{\frac{\pi}{2}} \begin{pmatrix} \overset{1}{\cos^2(t)+\sin^2(t)} \\ \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt$$

$$\int_0^{\frac{\pi}{2}} -\sin(t) \quad 2\cos(t)\,dt = \Big[\left(\cos(t) + 2\sin(t)\right)\Big]_0^{\frac{\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right) - \cos(0) - 2\sin(0)$$

$$= 0 + 2\cdot 1 - 1 - 0 = \underline{\underline{1}}$$

17| $f(x) = x^2 \cdot \ln(x) \quad \Rightarrow \quad \frac{df}{dx} = f'(x) = x^2 \cdot \frac{1}{x} + 2x \cdot \ln(x) = x + 2x\ln(x)$

$\Rightarrow df = x + 2x\ln(x)\,dx \qquad \Delta f = x_0 + 2x_0\ln(x_0) \quad \Delta x$

$x_0 = 2 \Rightarrow \quad f(2) = 4\cdot\ln(2) = 4\cdot 0{,}693 = 2{,}782$ Funktionswert

Differential in $x_0=2$:

$$df = (2 + 4\ln(2))\,dx$$
$$df = 4{,}782 \cdot dx$$
$$\Delta f = 4{,}782 \cdot \Delta x$$

$x = 1{,}9 \Rightarrow \Delta x = x - x_0 = 1{,}9 - 2 = -0{,}1$

$\Rightarrow \Delta f = 4{,}782 \cdot (-0{,}1) = -0{,}4782$

$\Rightarrow f(1{,}9) \approx 2{,}782 - 0{,}4782 = \underline{\underline{2{,}304}}$

$x = 2{,}2 \Rightarrow \Delta x = 0{,}2 \Rightarrow \Delta f = 4{,}782 \cdot 0{,}2 = 0{,}9564$

$\Rightarrow f(2{,}2) \approx 2{,}782 + 0{,}9564 = \underline{\underline{3{,}738}}$

$\underline{18|}$ a) $\int_1^2 x^2 \cdot \ln(x^3+1)\,dx = \frac{1}{3}\int_1^2 3x^2 \cdot \ln(x^3+1)\,dx$

$= \frac{1}{3} \cdot \int \ln(u)\,du = \frac{1}{3}\, u \cdot (\ln(u) - 1) + C$

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$= \frac{1}{3}\, (x^3+1) \cdot \left(\ln(x^3+1) - 1\right)\Big]_1^2$

$= \frac{1}{3} \cdot \left(9 \cdot (\ln(9) - 1) - 2 \cdot (\ln(2) - 1)\right) = \frac{1}{3} \cdot (10{,}775 + 0{,}614)$

$= \underline{\underline{3{,}79}}$

b) $\int (x^3 - 2)\, \ln(x)\, dx = 3x^2 \cdot \ln(x) - \cancel{\int 3x^2 \cdot \frac{1}{x}\, dx}$

v' u

$\cancel{= 3x^2 \ln(x) - \int 3x\, dx}$

$\cancel{= 3x^2 \ln(x) - \frac{3}{2}x^2 + C}$

$v = \cancel{3x^2}\ \frac{x^4}{4} - 2x \qquad u' = \frac{1}{x}$

$= \left(\frac{x^4}{4} - 2x\right) \ln(x) - \int \left(\frac{x^4}{4} - 2x\right) \cdot \frac{1}{x}\, dx$

$= \left(\frac{x^4}{4} - 2x\right) \ln(x) - \int \frac{x^3}{4} - 2\, dx$

$= \underline{\underline{\left(\frac{x^4}{4} - 2x\right) \ln(x) - \frac{x^4}{16} + 2x + C}}$

18 c) $\int \frac{x^3-3x+1}{(x-1)^2 \cdot x}\,dx$ -P11

$(x-1)^2 \cdot x = (x^2-2x+1)\cdot x$
$= x^3-2x^2+x$

Polynomdiv.:

$x^3-3x+1 : x^3-2x^2+x = 1$
$\underline{(x^3-2x^2+x)}$
$2x^2-4x+1$

$\Rightarrow \frac{x^3-3x+1}{(x-1)^2 \cdot} = 1 + \frac{2x^2-4x+1}{(x-1)^2 \cdot x}$

Partialbruchzerlegung von ↑

$\frac{2x^2-4x+1}{(x-1)^2 \cdot x} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}$

$2x^2-4x+1 = A_1(x-1)^2 + A_2 x\cdot(x-1) + A_3\, x$

$x=0$: $1 = A_1$
$=1$: $-1 = A_3$
$x=$: $1 = A_1+2A_2+A_3 = 1+2A_2-2 \Rightarrow A_2 = 1$

$\Rightarrow \int \frac{x^3-3x+1}{(x-1)^2}\,dx = \int 1 + \frac{1}{x} + \frac{1}{x-1} - \frac{1}{(x-1)^2}\,dx$

$= x + \ln|x| + \ln|x-1| + \frac{1}{x-1} + c$

19|

$$\lim_{x \to 0} \frac{\overset{\nearrow 0}{x^3}}{\underset{\searrow 0}{\sin(x)-x}} = \lim_{x \to 0} \frac{3x^2}{\underset{\searrow 0}{\cos(x)-1}} = \lim_{x \to 0} \frac{6x}{\underset{\searrow 0}{-\sin(x)}}$$

-P12

$$= \lim_{x \to 0} \frac{6}{-\cos(x)} \quad \frac{6}{-1} = \underline{\underline{-6}}$$

20| $v(t) = v_0 e^{-0{,}2t}$ $v(0) = v_0$ ✓

a) $S(t) = \int v_0 e^{-0{,}2t} dt = v_0 \frac{1}{-0{,}2} \int -0{,}2\, e^{-0{,}2t} dt$

$u = -0{,}2t$

$\frac{du}{dt} = u' = -0{,}2$

$du = -0{,}2 dt$

$= \frac{v_0}{-0{,}2} \int e^u du = \frac{v_0}{-0{,}2} e^{-0{,}2t} + c$

$= -5 v_0 e^{-0{,}2t} + c$

$S(0) = 0 \Rightarrow -5v_0 + c = 0 \Rightarrow \underline{\underline{c = 5v_0}}$

b) $a(t) = s'(t)$

$a(t) = -0{,}2\, v_0\, e^{-0{,}2t}$

c) $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} -5v_0 e^{-0{,}2t} + 5v_0 = 5v_0$

Das Fahrzeug stoppt also bei $s = 5v_0$

21| $w = 6$

Gleichungen:

Ca: $10 \cdot x = u + v \Rightarrow 10x - u - v = 0$

F $2x = \underset{6}{\underset{\shortparallel}{w}} \Rightarrow \underline{\underline{x = 3}}$

P: $6x = 2u \Rightarrow \underline{\underline{u = 9}}$

O: $24\underset{3}{\underset{\shortparallel}{x}} + 4y = 8\underset{9}{\underset{\shortparallel}{u}} + 4v \Rightarrow y = v$ $\quad \underline{\underline{v = 12}}$

H $2y = 2\underset{9}{\underset{\shortparallel}{u}} + \underset{6}{\underset{\shortparallel}{w}} \rightarrow 2y = 24 \Rightarrow \underline{\underline{y = 12}}$

S: $y = v$ (s.o.)

$\Rightarrow$ 3 Anteile $Ca_{10}F_2P_6O_{24}$

12 Anteile H_2SO_4

9 Anteile $CaH_2P_2O_8$

12 Anteile $CaSO_4$

22|

$$\sum_{i=10}^{17} 2^i = \sum_{i=0}^{17} 2^i - \sum_{i=0}^{9} 2^i$$

$$= \frac{2^{18}-1}{2-1} - \frac{2^{10}-1}{2-1} = 2^{18} - 2^{10}$$

23|

$$p(t) = \ln(\alpha t) - t = \ln(\alpha) + \ln(t) - t$$

$$p'(t) = \frac{1}{t} - 1 \overset{!}{=} 0 \Rightarrow \frac{1}{t} - 1 = 0 \Rightarrow \frac{1}{t} = 1 \Rightarrow \underline{\underline{t = 1}}$$

$$p''(t) = -\frac{1}{t^2} \Rightarrow p''(1) = -1 < 0 \Rightarrow \text{Maximum}$$

Wert: $p(1) = \ln(\alpha)$